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A formula for the hopping expansion of 8-vertex models coupled to an external field

Christof Gattringer

Massachusetts Institute of Technology
Center for Theoretical Physics
77 Massachusetts Avenue
Cambridge MA 02139, USA

Abstract

We study a generalized 8-vertex model where the vertices are coupled to a locally varying field. We rewrite the partition function as an integral over Grassmann variables. In this form it is possible to explicitly evaluate all terms of the hopping expansion. Applications of the resulting formula, in particular its relation to 2-D lattice field theories with fermions are discussed.

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1 Introduction

Rewriting physical systems in different representations is a powerful tool in theoretical physics. Alternative representations of a system can stress different aspects and allow new insights into its physical behavior. Sometimes a model is more accessible to numerical methods in a rewritten form or can even be solved explicitly.

Here we study a generalized version of the 8-vertex model [1, 2, 3], where now the vertices couple to an external field. This generalized model encompasses a large class of physically interesting systems such as spin models with locally varying coupling, polymers in an external field and in particular 2-D lattice field theories with fermions. The model first will be represented as an integral over Grassmann variables and in this representation one then can explicitly evaluate all terms of the hopping expansion. The resulting expression is a new representation for the partition function of the original model in terms of loops. In this new form it is possible to make connection to 2-dimensional lattice field theories with Wilson fermions [4]. The formula e.g. can be used to considerably simplify the hopping expansion of the lattice fermion determinant in an external scalar field, since it establishes that a large class of contributions (all terms with multiply occupied links) gets cancelled. Furthermore, our formula allows to explicitly integrate out the external field and to obtain simple loop representations for e.g. the Gross-Neveu model (see Section 5 for a brief discussion of these applications).

The article is organized as follows. In the next section we introduce the model and discuss its use for polymer systems and 2-D statistical models. Section 3 contains the reformulation of the partition function as an integral over Grassmann variables and the hopping expansion. In Section 4 the traces over the hopping generators are computed and the final expression for the hopping expansion is presented. The article closes with a brief discussion of the properties and applications of our expansion formula.

2 Definition of the generalized 8-vertex model

We analyze a generalized version of the 8-vertex model where now the vertices are coupled to a locally varying external field $B_\mu(x)$. The standard 8-vertex model [1, 2, 3] can be viewed as a model of 8 quadratic tiles (vertices) and each of them is assigned a weight w_i ($i = 1, ...8$) (compare Fig. 1).

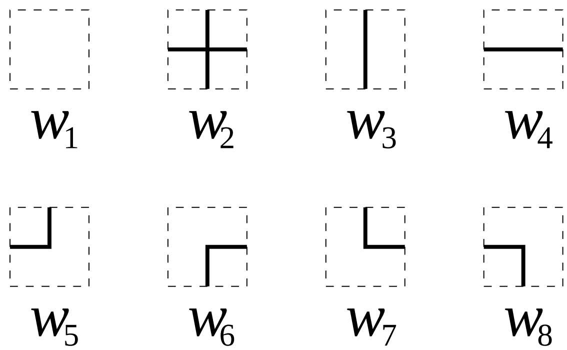

Figure 1: The eight vertices (tiles) and their weights w_i.

Consider now a lattice Λ which we assume to be a finite, rectangular piece of $\mathbb{Z}^2$ (the generalization to e.g. a torus is straightforward). A *tiling* of this lattice is a covering of Λ with the tiles such that on each site of Λ we place one of our tiles with the centers of the tiles sitting on the sites. The set $\mathcal{T}$ of *admissible tilings* is given by those arrangements of tiles where the black lines on the tiles never have an open end (for sites at the boundary this implies that not all 8 tiles can be used there). The partition function of the standard 8-vertex model is the sum over all admissible tilings $t \in \mathcal{T}$ and the Boltzmann weight for a particular tiling t is given by the product of the weights w_i for all tiles used in the tiling t.

In our generalization of the model we now add an additional, local structure. To each link (x, μ), $\mu = 1, 2$ of Λ we assign a real- or complex-valued field $B_\mu(x)$, $\mu = 1, 2$. The *contour* $C(t)$ of a tiling t is defined to be the set of all links of Λ which are occupied by black lines from the tiles. Since we allow only admissible tilings, each link is either occupied or empty and 'half-occupied' links do not occur. The partition function of the generalized model is now given by

$$Z \;=\; \sum_{t \in \mathcal{T}} \; \prod_{i=1}^{8} {w_i}^{n_i(t)} \prod_{(x,\mu) \in C(t)} B_\mu(x) \; . \tag{1}$$

By $n_i(t)$ we denote the abundance of tile Nr. i in a given tiling t. Note that the contour $C(t)$ is simply the set of links occupied by black lines and does not have an orientation. So in case the field B_μ is chosen complex, a link variable $B_\mu(x)$ is always counted as it is, and no complex conjugation is implied.

A second (less general) form of the model is obtained by assigning a scalar field $\varphi(x)$ to all sites of Λ and setting

$$B_\mu(x) \; = \; \sqrt{\varphi(x)}\sqrt{\varphi(x+\hat{\mu})} \; ,$$

where $\hat{\mu}$ denotes the unit vector in direction μ. In case one of the φ is negative B_μ becomes complex, a case which we explicitly included above. The partition function now reads

$$Z \; = \; \sum_{t \in \mathcal{T}} \prod_{i=1}^{8} {w_i}^{n_i(t)} \prod_{x \in S(t)} \varphi(x) \; . \tag{2}$$

Here $S(t)$ denotes the set of all sites occupied by the tiling t. When a site x is occupied by tile Nr. 2, this site is counted twice giving a factor $\varphi(x)^2$. In case x is occupied by tile Nr. 1, $x \notin S(t)$ and the factor is 1. For all other tiles x is counted once and the factor is $\varphi(x)$.

The generalized 8-vertex model in its two forms (1), (2) encompasses several interesting physical systems (for a detailed discussion of the corresponding models at trivial external field see [3] and references therein). For example by setting the parameters in formulation (1) to $w_1 = w_2 = ... = w_8 = 1$ and the external fields to $B_\mu(x) = e^{-2J_\mu(x)/kT}$, the model describes the Ising model with locally varying coupling $J_\mu(x)$ for links (x,μ). The set $C(t)$ has the interpretation of a Peierls contour which separates patches of up and down spins. Ising type models with next to nearest neighbor terms can be obtained by chosing different values for the weights w_i. The model with locally varying couplings can furthermore be viewed as a generating functional for $2n$-point functions in Ising type models: Differentiating Z in the form (1) with respect to the link variables on a connected path produces the 2-point function for two spins sitting at the endpoints of the path (for $2n$-point functions use a net of paths).

In its form (2), the generalized vertex model can be used to describe the physics of loop gases and polymers (compare e.g. [5, 6, 7, 8, 9, 10]) in an external field φ.

Most remarkable, however, is the fact that our model is related to 2-D lattice field theories with Wilson fermions. For staggered fermions it has long been known, that a polymer representation can be found [11, 12, 13, 14]. Due to their spinor structure, a similar map for Wilson fermions [15] is more complicated. Based on the hopping expansion which will be discussed

in the next section, it was shown in [4] that 2-D Wilson fermions in a scalar background field $\varphi(x)$ are equivalent to our model in its form (2) (see Section 5 for a brief discussion of the argument used in [4]).

3 Grassmann representation and hopping expansion

The first step in computing the hopping expansion is finding a representation of the partition function through a Grassmann integral. For the standard 8-vertex model this problem has been solved [16, 17, 18]. The central idea is to assign to each site x of the lattice 4 Grassmann variables $\eta_{+1}(x), \eta_{-1}(x), \eta_{+2}(x), \eta_{-2}(x)$ and integrate a Boltzmann factor with an appropriately chosen action $S[\eta]$ to obtain

$$Z = (-1)^{|\Lambda|} \int \prod_{x \in \Lambda} d\eta_{-2}(x) d\eta_{+2}(x) \; d\eta_{-1}(x) d\eta_{+1}(x) e^{S[\eta]}. \tag{3}$$

Here $|\Lambda|$ denotes the size of the lattice. The action is a quadratic form in the η_i and contains two types of terms. Hopping terms which are products of Grassmann variables at neighboring sites create link elements, i.e. the black lines on our tiles. In addition, quadratic terms based on only one site are needed to saturate the Grassmann integral. These terms enable the different moves the lines perform on the different tiles, e.g. a corner on tiles 5-8, or keep going straight on tiles 3,4. For an explicit discussion of the Grassmann representation for the standard 8-vertex model see [16].

For the generalized model in the external field $B_\mu(x)$ only the hopping terms have to be modified such that the links are furnished with their corresponding fields. The action $S = S_h + S_m + S_c$ producing the generalized 8-vertex model then consists of hopping, monomer and corner terms

$$\begin{aligned}
S_h &= \sqrt{w_2} \sum_{x \in \Lambda} \Big[B_1(x) \; \eta_{+1}(x) \eta_{-1}(x + \hat{1}) \\
&\qquad + B_2(x) \; \eta_{+2}(x) \eta_{-2}(x + \hat{2}) \Big] \, , \\
S_m &= -\frac{1}{\sqrt{w_2}} \sum_{x \in \Lambda} \Big[w_3 \eta_{-1}(x) \eta_{+1}(x) + w_4 \eta_{-2}(x) \eta_{+2}(x) \Big], \\
S_c &= -\frac{1}{\sqrt{w_2}} \sum_{x \in \Lambda} \Big[w_5 \; \eta_{+1}(x) \eta_{-2}(x) + w_6 \; \eta_{+2}(x) \eta_{-1}(x) \\
&\qquad + w_7 \; \eta_{-2}(x) \eta_{-1}(x) + w_8 \; \eta_{+2}(x) \eta_{+1}(x) \Big] \, .
\end{aligned} \tag{4}$$

The boundary conditions are open, i.e. hopping terms that would lead out of our rectangular lattice are omitted. Inserting (4) into the path integral, expanding the exponent termwise and integrating the fields along the lines of [16] establishes the representation (3) for the partition function (1).

We remark, that the action (4) is independent of w_1. However, it turns out, that for the Grassmann representation (3), (4) with a quadratic action to work, the weights w_i have to obey the free fermion condition [1]

$$w_1 w_2 \;+\; w_3 w_4 \;=\; w_5 w_6 \;+\; w_7 w_8 \;. \tag{5}$$

Thus w_1 can be computed from the independent parameters $w_i, i > 1$. Note that for the case of a *trivial external field* (all $B_\mu(x) = 1$), the free fermion condition is a sufficient condition for finding an explicit solution. This is most easily seen by first constructing the Grassmann representation (which always can be done when (5) holds) and then performing a Fourier transformation of the Grassmann variables. The Fourier transformation diagonalizes the action and the free energy can be readily computed. However, when the fields $B_\mu(x)$ are chosen non-trivially, Fourier transformation produces a convolution of the external field and does no longer diagonalize the action.

Here we analyze the hopping expansion for the generalized 8-vertex model. The next step is to anti-symmetrize the action by subtracting from the above expression (4) the same terms but with reversed order of the Grassmann variables and dividing by 2. It is convenient to denote the resulting action using matrix notation. We order the Grassmann variables in a vector (T denotes transposition)

$$\eta(x) \;=\; \Big(\eta_{+1}(x), \eta_{-1}(x), \eta_{+2}(x), \eta_{-2}(x)\Big)^T \;, \tag{6}$$

and define the 4×4 matrices $P_{\pm 1}$ and $P_{\pm 2}$

$$\begin{array}{lcl} P_{+1}(i,j) \equiv \sqrt{w_2}\,\delta_{i,1}\delta_{j,2} & , & P_{-1}(i,j) \equiv -\sqrt{w_2}\,\delta_{i,2}\delta_{j,1}, \\ P_{+2}(i,j) \equiv \sqrt{w_2}\,\delta_{i,3}\delta_{j,4} & , & P_{-2}(i,j) \equiv -\sqrt{w_2}\,\delta_{i,4}\delta_{j,3}. \end{array} \tag{7}$$

They obey $P_\nu^T = -P_{-\nu}$. We also define

$$\mu \;=\; \frac{1}{\sqrt{w_2}} \left(\begin{array}{cccc} 0 & -w_3 & -w_8 & +w_5 \\ +w_3 & 0 & -w_6 & -w_7 \\ +w_8 & +w_6 & 0 & -w_4 \\ -w_5 & +w_7 & +w_4 & 0 \end{array} \right) \;,$$

with

$$\mu^{-1} \;=\; \frac{\sqrt{w_2}}{w_1 w_2}\begin{pmatrix} 0 & -w_4 & +w_7 & -w_6 \\ +w_4 & 0 & +w_5 & +w_8 \\ -w_7 & -w_5 & 0 & -w_3 \\ +w_6 & -w_8 & +w_3 & 0 \end{pmatrix} .$$

We remark that for writing μ^{-1} in this form the free fermion condition (5) was used. The determinant of μ is given by $\det \mu = w_1^2$ and $\mu^T = -\mu$.

Using the matrix notation the action reads

$$S[\eta] \;=\; \frac{1}{2}\sum_{x,y} \eta(x)^T K(x,y)\eta(y) \;,$$

with the kernel K consisting of 2 parts $K = -M + R$, where

$$M(x,y) = \mu\; \delta_{x,y} \;\;,\;\; R(x,y) = \sum_{\nu=\pm 1}^{\pm 2} B_\nu(x) P_\nu\; \delta_{x+\hat{\nu},y} \;,$$

and we have defined

$$B_{-\nu}(x) \;=\; B_\nu(x-\hat{\nu}) \;,\quad \nu \;=\; 1,2 \;.$$

It has to be stressed, that here the link variable is *not* complex conjugated when hopping in negative ν-direction. This is different from lattice gauge theory (see e.g. [19]) where the link variables are conjugated for backward hopping. This difference indicates that for lattice field theories with fermions coupled to a vector field a different structure emerges.

The partition function can now be written as a Pfaffian, and since by construction the kernel K of the action is anti-symmetric, the Pfaffian is given by the root of the determinant of K. An overall factor $\det[-M] = w_1^2$ can be extracted and the rest is expanded in powers of the hopping matrix giving

$$\begin{aligned} Z \;&=\; (-1)^{|\Lambda|} \int D\eta\; e^{\frac{1}{2}\eta^T K \eta} \;=\; (-1)^{|\Lambda|}\; \mathrm{Pf}\, K \\ &=\; (-1)^{|\Lambda|}\; \sqrt{\det[-M+R]} \\ &=\; (-1)^{|\Lambda|}\sqrt{\det[-M]}\sqrt{\det[1-M^{-1}R]} \\ &=\; (-1)^{|\Lambda|} w_1^{|\Lambda|}\; e^{-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr} H^n} , \end{aligned} \tag{8}$$

where we introduced the hopping matrix

$$H(x,y) \;=\; M^{-1}R(x,y) \;=\; \frac{1}{w_1}\sum_{\nu=\pm1}^{\pm2} B_\nu(x)\Gamma_\nu \, \delta_{x+\hat{\nu},y} \; , \tag{9}$$

with $\Gamma_\nu \;=\; w_1\mu^{-1}P_\nu$, or explicitly

$$\Gamma_{+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & +w_4 & 0 & 0 \\ 0 & -w_7 & 0 & 0 \\ 0 & +w_6 & 0 & 0 \end{pmatrix} \quad , \quad \Gamma_{-1} = \begin{pmatrix} +w_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +w_5 & 0 & 0 & 0 \\ +w_8 & 0 & 0 & 0 \end{pmatrix} \; ,$$

$$\Gamma_{+2} = \begin{pmatrix} 0 & 0 & 0 & +w_7 \\ 0 & 0 & 0 & +w_5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +w_3 \end{pmatrix} \quad , \quad \Gamma_{-2} = \begin{pmatrix} 0 & 0 & +w_6 & 0 \\ 0 & 0 & -w_8 & 0 \\ 0 & 0 & +w_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; .$$

The series in the exponent of (8) converges for $||H|| < 1$. By choosing suitable values for the weights w_i and the fields $B_\mu(x)$ this can be achieved. Note however, that this restriction is only a technical assumption due to the expansion used here. The final result is simply a finite polynomial in the w_i and $B_\mu(x)$ and the restriction of the variables can be lifted then.

Due to the Kronecker deltas in (9) the traces $\mathrm{Tr}H^n$ reduce to products of matrices Γ_ν supported on closed loops l. Since on the rectangular lattice all closed loops have even length, traces over odd powers of H vanish. For even powers we obtain

$$\mathrm{Tr}\; H^{2k} \;=\; \left(\frac{1}{w_1}\right)^{2k} \sum_{x\in\Lambda} \sum_{l\in\mathcal{L}_x^{(2k)}} \prod_{(y,\nu)\in l} B_\nu(y) \; \mathrm{Tr} \prod_{\mu\in l} \Gamma_\mu \; . \tag{10}$$

By $\mathcal{L}_x^{(2k)}$ we denote the set of all closed, connected loops of length $2k$ and base point x. The first product picks up a factor $B_\nu(y)$ whenever the loop $l \in \mathcal{L}_x^{(2k)}$ runs through the link (y,ν). The last term is simply the trace of the product of matrices Γ_μ as they appear along the loop. By explicit calculation one can show that

$$\Gamma_{\pm\mu}\,\Gamma_{\mp\mu} \;=\; 0 \; , \qquad \mu = 1,2 \; .$$

This implies that whenever the loop l turns around at a site and runs back on its last link the contribution of this *back-tracking* loop vanishes. Thus the

set $\mathcal{L}_x^{(2k)}$ only contains closed, connected, non back-tracking loops of length $2k$ with base point x. For each loop both possible orientations have to be taken into account. The loops in $\mathcal{L}_x^{(2k)}$ can self-intersect and retrace parts of their contour or even their whole contour.

4 Computing the traces and final result

In this section we evaluate the traces over the matrices Γ_μ. The loops $l \in \mathcal{L}_x^{(2k)}$ are decomposed into 4 simple elements (depicted in Fig. 2) and algebraic relations for products of Γ_μ corresponding to these elements can be used to give a constructive procedure for determining the traces $\text{Tr}\prod_{\mu\in l}\Gamma_\mu$ (compare also [20] where the case of the Ising model in this expansion was discussed in more detail).

The first of the four elements, *basic loop*, is the closed loop around a single plaquette (compare Fig. 2.a). One finds by direct evaluation

$$\text{Tr}\,\Gamma_\mu\Gamma_\nu\Gamma_{-\mu}\Gamma_{-\nu} \;=\; -w_5w_6w_7w_8\;, \tag{11}$$

where $\mu,\nu = \pm1,\pm2\;,\;\mu\neq\pm\nu$. Thus for the basic loop around a single plaquette we always obtain -1 times the product of the weights for the 4 tiles showing a corner (compare Fig. 1), independent of the starting point and the orientation.

The second element is referred to as *telescope rule*. The telescope rule can also be shown by explicit evaluation and reads

$$\Gamma_{\pm1}\Gamma_{\pm1} \;=\; w_4\Gamma_{\pm1} \quad , \quad \Gamma_{\pm2}\Gamma_{\pm2} \;=\; w_3\Gamma_{\pm2}\;.$$

The geometrical interpretation of these two identities is simple. A piece of loop can be shrunk or stretched in horizontal (vertical) direction and a factor w_4 (w_3) has to be collected (compare the graphical representation in Fig. 2.b).

The third rule is the *kink rule* depicted in Fig. 2.c. Algebraically it reads

$$\begin{aligned}\Gamma_{\pm\mu}\Gamma_{\pm\nu}\Gamma_{\pm\mu} &= w_5w_6\;\Gamma_{\pm\nu}\;,\quad \mu,\nu=1,2\;,\;\mu\neq\nu\;,\\ \Gamma_{\pm\mu}\Gamma_{\mp\nu}\Gamma_{\pm\mu} &= w_7w_8\;\Gamma_{\pm\nu}\;,\quad \mu,\nu=1,2\;,\;\mu\neq\nu\;.\end{aligned}$$

This rule, together with the telescope rule allows to remove superfluous kinks or corners in a loop (or sub-loop) and to reduce this loop (sub-loop) to rectangular form which then can be shrunk to a loop (sub-loop) around a single

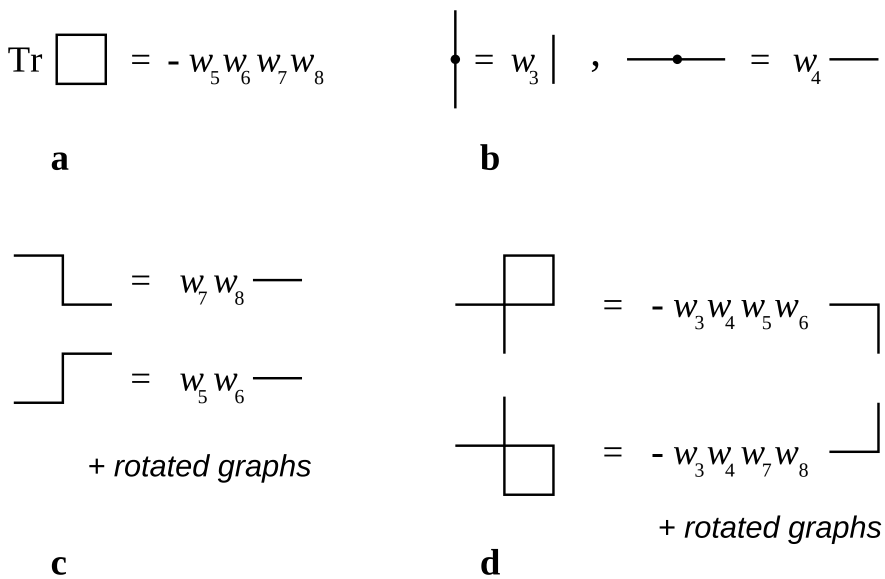

Figure 2: The four elements used for the computation of the traces: a: basic loop, b: telescope rule, c: kink rule, d: intersection rule.

plaquette using the telescope rule. One simply replaces a piece of loop by a simpler piece (i.e. one with less corners) which can be constructed using telescope and kink rule (see [20] for a detailed description of the procedure). During this process one has to collect all emerging factors w_i. In case the loop we started with contained no sub-loop, this procedure will reduce it to the basic loop and using (11) the result for the trace can be read off.

In case the loop contains sub-loops, we first reduce a sub-loop which has only one self-intersection (such a loop always exists) to a sub-loop around a single plaquette. The sub-loop can then be removed by the fourth rule, the *intersection rule*. It states, that when removing a sub-loop, in addition to the corner and monomer factors, a factor of -1 has to be collected. The algebraic expression reads (for the graphical representation see Fig. 2.d)

$$\begin{aligned} \Gamma_{\pm\mu}\Gamma_{\pm\mu}\Gamma_{\pm\nu}\Gamma_{\mp\mu}\Gamma_{\mp\nu}\Gamma_{\mp\nu} &= -w_3 w_4 w_5 w_6 \; \Gamma_{\pm\mu}\Gamma_{\mp\nu} \; , \\ \Gamma_{\pm\mu}\Gamma_{\pm\mu}\Gamma_{\mp\nu}\Gamma_{\mp\mu}\Gamma_{\pm\nu}\Gamma_{\pm\nu} &= -w_3 w_4 w_7 w_8 \; \Gamma_{\pm\mu}\Gamma_{\pm\nu} \; , \end{aligned}$$

for $\mu, \nu = 1, 2$, $\mu \neq \nu$. As outlined above, and discussed in more detail in [20], the four elements can be combined to compute the trace in a construc-

tive way. After collecting all monomer, corner and intersection factors, we find the result

$$\mathrm{Tr} \prod_{\mu \in l} \Gamma_\mu \;=\; -(-1)^{s(l)} \prod_{i=3}^{8} w_i^{n_i(l)} \; . \tag{12}$$

Here $s(l)$ denotes the number of self-intersections the loop l has. The exponents $n_i(l)$ give the numbers for the abundance of the line elements as they are depicted in Fig. 1. E.g. when the loop changes from heading east to heading north at a site, it picks up a factor of w_5, and similarly for the other tiles. Note that the loops l in the hopping expansion appear as an ordered set of instructions for the directions the loop takes as it hops from one site to the next. The element corresponding to tile Nr. 2 with weight w_2 (compare Fig. 1) does not appear. We also remark, that the result (12) is independent of the orientation of the loop.

Inserting (12) and (10) into (8) one finds

$$Z = (-1)^{|\Lambda|} w_1^{|\Lambda|} \; \exp\left(\frac{1}{2} \sum_{k=1}^{\infty} \frac{{w_1}^{-2k}}{2k} \sum_x \sum_{l \in \mathcal{L}_x^{(2k)}} (-1)^{s(l)} \prod_{(y,\nu) \in l} B_\nu(y) \; \prod_{i=3}^{8} w_i^{ni(l)} \right). \tag{13}$$

The final step is to eliminate the explicit summation over the base points. When doing so a minor subtlety has to be discussed. Consider first a loop which runs through its contour only once. Each of the $2k$ lattice points it visits can serve as the base point producing a factor $2k$ which cancels the corresponding factor in (13). If now a loop l completely iterates its contour $I(l)$-times, then there are only $2k/I(l)$ different choices for a base point and a factor $1/I(l)$ remains. The final result is

$$Z \;=\; (-1)^{|\Lambda|} w_1^{|\Lambda|} \times \exp\left(\sum_{l \in \mathcal{L}} \frac{(-1)^{s(l)}}{I(l)} \Big(\frac{1}{w_1}\Big)^{|l|} \prod_{(x,\nu) \in l} B_\nu(x) \; \prod_{i=3}^{8} w_i^{n_i(l)} \right) . \tag{14}$$

Here the sum runs over the set $\mathcal{L}$ of all closed, non back-tracking loops of arbitrary length. Each loop is included with only one of its two possible orientations leading to the cancellation of the factor 1/2 which appeared in the last expression. By $|l|$ we denote the length of the loop l and $I(l)$ is the number of iterations of its complete contour. We remark, that (14) does not explicitly contain w_2, but this weight is related to the other weights through the free fermion condition (5).

5 Discussion of the result

Several aspects of the result (14) should be discussed. When viewing vertex models as models of Peierls contours for Ising-type spin models the exponent in (14) is proportional to the free energy in terms of loops. For the standard Ising model a representation of this type is given in [21]. Equation (14) generalizes this results to 8-vertex models coupled to an external field which also includes the case of $\mathbb{Z}_2$-spin models with locally varying couplings.

The expression (14) is also remarkable from an algebraic point of view. In principle, it is possible to expand the exponent in (14) in a power series of ${w_1}^{-2k}$. The coefficients in this series can be expressed in terms of products of loops $l \in \mathcal{L}$. In turn one would like to identify these loops with the contours $C(t)$ which occur for the tilings t in the original formulation of the model. The loops $l \in \mathcal{L}$, however, can occupy links several times and also taking products of these loops will create multiply occupied links while the contours $C(t)$ only have empty or singly occupied links. When analyzing some terms of this expansion, one finds however, that due to the self-intersection factor $(-1)^{s(l)}$ all contributions where links are multiply occupied get cancelled. Only terms without multiply occupied links survive and these correspond to the contors $C(t)$. Our equation (14) establishes a remarkable feature of this expansion, namely that the cancellation of terms with multiply occupied links is independent of the corner and monomer weights. When applying the formula to fermionic lattice field theories (see below) this cancellation of all terms with multiply occupied links implies a considerable simplification of the hopping expansion for the fermion determinant.

Probably the most interesting application of Eq. (14) is its use when relating fermionic lattice field theories to vertex models. An example of this type of application of our formula Eq. (14) is discussed in [4], where it is proven, that 2-D Wilson fermions in an external scalar field $\varphi(x)$ are equivalent to our model in its form (2) with $w_1 = 1, w_2 = 0, w_3 = w_4 = \kappa, w_4 = ... \; w_8 = \kappa/\sqrt{2}$, where κ is related to the mass parameter through $\kappa = (2+m)^{-1}$. For the case of free Wilson fermions ($\varphi(x) = 1$) this was first shown by Scharnhorst [23] with different methods (for a presentation using the techniques of this paper see [24]). [4] extends this result to non-trivial background fields using a careful analysis of the hopping expansion for the Wilson fermions. Wilson fermions give rise to a bilinear Grassmann action (instead of a quadratic form here) and the object to be expanded is the

fermion determinant (instead of the Pfaffian). The Dirac γ-matrices play the role of the hopping generators Γ_μ which we encounter here, and also for the Wilson fermions it is possible to compute all the traces which appear in the hopping expansion. The final step in the proof of equivalence is the identification of the correct weights w_i by comparing the hopping expansions for the two models. The details of this mapping are given in [4]. Further results are obtained in [4] by integrating out the scalar field, which can e.g. be used to generate the Gross-Neveu model and map it to another vertex model (for similar mappings of the Schwinger model and the 2-D Thirring model with Wilson fermions see [23, 25, 26]). Such mappings of lattice field theories onto vertex models (loop representation) are powerful tools, since often the numerical simulation of the vertex model is simpler than analyzing the original model and allows for higher precision (see e.g. [27] where this was demonstrated for the case of the strongly coupled Schwinger model). In particular the loop expansion of the fermion determinant developed here and in [4] overcomes the fermion sign problem. This permutation sign plagues fermionic systems in the Hamiltonian approach [28] as well as in the Grassmann path integral formulation [29] and in a numerical treatment requires an exponentially increasing number of Monte Carlo configurations as the volume or inverse temperature are increased [28]. Currently we are working on an implementation of new numerical algorithms which are based directly on the loop representation [30].

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